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On Dimension and Existence of Local Bases for Multivariate Spline Spaces

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We consider spaces of splines in k variables of smoothness r and degree d defined on a polytope in $\mathbb{R}^k$ which has been divided into simplices. Bernstein–Bézier methods are used to develop a framework for analyzing dimension and basis questions. Dimension formulae and local bases are found for the case $r=0$ and general k. The main result of the paper shows the existence of local bases for spaces of trivariate splines (where $k=3$) whenever $d>8r$.

1. INTRODUCTION

We begin by defining a *triangulation* in $\mathbb{R}^k$. Suppose that $k \geqslant 1$ and that $\mathscr{V} \subset \mathbb{R}^k$ is a set of N distinct points. The following definitions are standard (see [3, 13]):

DEFINITION 1. A *κ-simplex* σ $(0 \leqslant \kappa \leqslant k)$ is the convex hull of $\kappa+1$ points called the vertices of σ. σ is *non-degenerate* if its κ-dimensional

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volume is non-zero and *degenerate* otherwise. The *dimension* of a non-degenerate κ-simplex is κ. The set of vertices of σ is denoted by $\langle\sigma\rangle$. The convex hull of a subset of $\langle\sigma\rangle$ containing $\mu+1 \leqslant \kappa$ elements is a *μ-face* of σ. A $(\kappa-1)$-face of σ is also called a *facet of* σ. The convex hull of a finite set P of points is denoted by $\operatorname{conv}(P)$.

DEFINITION 2. A *triangulation* $\mathscr{T}$ of the set $\mathscr{V}$ is a set of non-degenerate k-simplices satisfying the following requirements:

1. All vertices of each simplex in $\mathscr{T}$ are elements of $\mathscr{V}$.
2. The interiors of the simplices in $\mathscr{T}$ are pairwise disjoint.
3. The simplices cover the convex hull of $\mathscr{V}$, i.e., as point sets we have

$$\Omega := \operatorname{conv}(\mathscr{V}) = \bigcup_{T \in \mathscr{T}} T. \tag{1}$$

4. Each facet of a simplex in $\mathscr{T}$ either is on the boundary of Ω or else is a common face of exactly two simplices in $\mathscr{T}$.
5. Each simplex in $\mathscr{T}$ contains no points in $\mathscr{V}$ other than its vertices.

Note that a μ-face of a simplex is itself a μ-dimensional simplex. We denote by $\mathscr{S}_\mu$ the set of all μ-faces of the simplices in $\mathscr{T}$ $(\mu = 0, 1, ..., k-1)$. We denote the set of all simplices (of various dimensions) by

$$\mathscr{S} = \bigcup_{\mu=0}^{k-1} \mathscr{S}_\mu \cup \mathscr{T}. \tag{2}$$

In the bivariate $(k=2)$ case $\mathscr{S}$ consists of vertices, edges, and triangles. In the trivariate $(k=3)$ case, $\mathscr{S}$ consists of vertices, edges, triangles and tetrahedra.

DEFINITION 3. The *star* of a simplex $\sigma \in \mathscr{S}$ is the point set

$$\operatorname{star}(\sigma) = \bigcup_{\tau \in \mathscr{T},\, \sigma \subset \tau} \tau. \tag{3}$$

Note that throughout this paper the symbol $\subset$ does not exclude equality. Thus the star of a simplex $\sigma \in \mathscr{T}$ is σ itself.

We employ the usual definition of binomial coefficients:

$$\binom{m}{n} = \begin{cases} \dfrac{m!}{n!\,(m-n)!}, & \text{if } m \geqslant n \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

DEFINITION 4. Given a triangulation $\mathscr{T}$ and integers $0 \leqslant r \leqslant d$, we define the corresponding *spline space*

$$S_d^r(\mathscr{T}) = \{s \in C^r(\Omega) : s|_\tau \in \mathbb{P}_d^k, \qquad \forall \tau \in \mathscr{T}\}, \tag{5}$$

where $\mathbb{P}_d^k$ is the $\binom{k+d}{d}$-dimensional linear space of all k-variate polynomials of total degree not exceeding d.

These spaces of splines are of considerable interest in numerical analysis and approximation theory, and have been studied heavily in the past 15 years (see [2, 4–6, 9–12, 14–17, 19, 20, 22] and references therein for the bivariate case and [1, 3, 7, 21] and references therein for the multivariate case).

Clearly $S_d^r(\mathscr{T})$ is a finite-dimensional linear space. One would like to calculate its dimension, and to construct a basis for it (preferably consisting of elements with small supports). But these problems are very difficult (and are not completely solved) even for the bivariate case, and so we cannot expect to be be able to resolve them completely (at this point in time) for the multivariate case.

Our aim in this paper is more modest. First, in Sections 2 and 3 we use Bernstein–Bézier methods to develop a framework in which the dimension and local basis problems can be analyzed. In Section 4 we give dimension formulae and explicit local bases for the special case of $r = 0$. Smoothness conditions are discussed in Section 5. The rest of the paper deals with trivariate splines where the triangulation becomes a tetrahedral partition. In Section 6 we give dimension formulae and explicit local bases for certain very special partitions. These results are used in Section 7 to establish the main result of the paper: the existence of local bases for trivariate spline spaces on arbitrary tetrahedral partitions whenever $d > 8r$.

2. The Generalized Bernstein–Bézier Form

Our analysis of $S_d^r(\mathscr{T})$ is based on the well-known (cf. [2, 8]) Bernstein–Bézier form of a multivariate polynomial. In this section we introduce the required notation. Let $\mathbb{N}$ be the set of non-negative integers. We shall use the set of vertices $\mathscr{V}$ of the triangulation as an index set. For vectors $\mathbf{i} = [i_V]_{V \in \mathscr{V}} \in \mathbb{N}^N$ and $\mathbf{a} = [a_V]_{V \in \mathscr{V}} \in \mathbb{R}^N$, let

$$|\mathbf{i}| = \sum_{V \in \mathscr{V}} i_V, \tag{6}$$

$$\mathbf{a}^{\mathbf{i}} = \frac{|\mathbf{i}|!}{\prod_{V \in \mathscr{V}} i_V!} \prod_{V \in \mathscr{V}} a_V^{i_V}, \tag{7}$$

where

$$0^0 := 1. \tag{8}$$

We also use the notation

$$\sigma(\mathbf{i}) = \operatorname{conv}\{V : i_V > 0\}, \qquad \sigma(\mathbf{a}) = \operatorname{conv}\{V : a_V \neq 0\}. \tag{9}$$

For each $V \in \mathscr{V}$, we define a *generalized barycentric coordinate* function as the cardinal piecewise linear function $b_V \in S_1^0(\mathscr{T})$ with the property

$$b_V(W) = \delta_{VW} = \begin{cases} 1, & \text{if } V = W \\ 0, & \text{otherwise,} \end{cases} \qquad \forall W \in \mathscr{V}. \tag{10}$$

Clearly, in each simplex $\tau \in \mathscr{T}$ the functions b_V, where V is a vertex of τ, reduce to the ordinary barycentric coordinates. Globally, i.e., for all $x \in \Omega$, they satisfy

$$\sum_{V \in \mathscr{V}} b_V = 1, \qquad b_V \geqslant 0 \qquad \forall V \in \mathscr{V}, \qquad \text{and} \qquad x = \sum_{V \in \mathscr{V}} b_V(x)\, V. \tag{11}$$

Associated with these coordinate functions, we define the vector-valued function

$$\mathbf{b}(x) = [b_V(x)]_{V \in V}. \tag{12}$$

It is known (cf. [2, 8]) that every function $s \in S_d^0(\mathscr{T})$ can be written uniquely as

$$s(x) = \sum_{\mathbf{i} \in I_d} c_{\mathbf{i}} \mathbf{b}^{\mathbf{i}}(x), \tag{13}$$

where $\mathbf{b}^{\mathbf{i}}$ is defined from $\mathbf{b}$ as in (7), and where I_d is the so-called *domain index set*

$$I_d = \{\mathbf{i} \in \mathbb{N}^N : |\mathbf{i}| = d \text{ and } \sigma(\mathbf{i}) \in \mathscr{S}\}. \tag{14}$$

The coefficients $c_{\mathbf{i}}$ are the *Bézier ordinates* of s. It will be useful to define a linear functional to pick off these coefficients. We define $\lambda_{\mathbf{i}} : S_d^0(\mathscr{T}) \to \mathbb{R}$ to be the functional defined by

$$\lambda_{\mathbf{i}}(s) = c_{\mathbf{i}}, \tag{15}$$

where $c_{\mathbf{i}}$ is the corresponding coefficient in the expansion (13).

For later use, we also introduce the set of *domain points*

$$P_{\mathbf{i}} = \sum_{V \in \mathscr{V}} \frac{i_V}{|\mathbf{i}|} V, \qquad \mathbf{i} \in I_d. \tag{16}$$

The pairs $(P_{\mathbf{i}}, c_{\mathbf{i}})$, $\mathbf{i} \in I_d$, are called the *Bézier control points* of s, and the set of all control points is called the *control net* of s.

We note that a domain point $P_{\mathbf{i}}$ lies at a vertex precisely when exactly one component of $\mathbf{i}$ is non-zero. More generally, $P_{\mathbf{i}}$ lies on a simplex σ of dimension q precisely when exactly $q+1$ components of $\mathbf{i}$ are non-zero (corresponding to the vertices of σ). For later use it will be convenient to define the *distance* of a point $P_{\mathbf{i}}$ from a simplex σ as

$$d(P_{\mathbf{i}}, \sigma) = \sum_{W \in \langle \sigma \rangle'} |\mathbf{i}_W|, \tag{17}$$

where

$$\langle \sigma \rangle' = \mathscr{V} \setminus \langle \sigma \rangle \tag{18}$$

is the set of all points in $\mathscr{V}$ which are not vertices of σ.

The following lemma will be useful later:

LEMMA 5.

$$|I_d| = \sum_{\sigma \in \mathscr{S}} \binom{d-1}{\dim \sigma}. \tag{19}$$

Proof. It suffices to show that each simplex σ contains exactly $\binom{d-1}{\dim \sigma}$ domain points that are not contained in any lower dimensional simplex. We now use induction on the dimension of the simplices. There is $1 = \binom{d-1}{0}$ domain point at each vetex. If $\kappa = \dim \sigma > 0$, the domain points not contained in a lower dimensional simples are precisely the domain points in the interior of σ. These can be obtained by stripping the $\kappa + 1$ facets of σ, leaving a κ-dimensional simplicial array of $\binom{d+\kappa-(\kappa+1)}{\kappa} = \binom{d-1}{\kappa}$ domain points. ∎

3. DETERMINING SETS

In the bivariate case (cf. [4–6, 12, 17]) the concept of a *determining set* proved to be very useful. We now introduce a multivariate analog.

DEFINITION 6. A set $\mathscr{D} \subset I_d$ is a *determining set* of $S_d^r(\mathscr{T})$ if, for all $s \in S_d^r(\mathscr{T})$,

$$c_{\mathbf{i}} = 0, \qquad \forall \mathbf{i} \in \mathscr{D} \qquad \Rightarrow \qquad s \equiv 0. \tag{20}$$

$\mathscr{D}$ is a *minimal determining set* if there is no determining set with fewer elements than $\mathscr{D}$.

The following lemma is an analog of Lemma 1 of [4], and can easily be established by elementary linear algebra.

LEMMA 7. *If $\mathscr{D}$ is a determining set for $S_d^r(\mathscr{T})$, then $|\mathscr{D}|$ is an upper bound on the dimension of $S_d^r(\mathscr{T})$. Moreover, if $\mathscr{D}$ is a minimal determining set, then $|\mathscr{D}|$ equals the dimension of $S_d^r(\mathscr{T})$.*

As shown in [4] for the bivariate case, minimal determining sets $\mathscr{D}$ can also be useful for constructing a basis (and dual basis) for the spline space. The same idea can be used here in the multivariate setting.

LEMMA 8. *Let $\mathscr{D}$ be a determining set for $S_d^r(\mathscr{T})$. Suppose that for each $\mathbf{i} \in \mathscr{D}$ there exists a spline $l_\mathbf{i}$ in $S_d^r(\mathscr{T})$ such that*

$$\lambda_\mathbf{j} l_\mathbf{i} = \delta_{\mathbf{ij}}, \qquad \forall \mathbf{j} \in \mathscr{D}. \tag{21}$$

Then $\mathscr{D}$ is a minimal determining set, the dimension of $S_d^r(\mathscr{T})$ is equal to $|\mathscr{D}|$, and the functions $\{l_\mathbf{i}\}_{\mathbf{i} \in \mathscr{D}}$ form a basis for $S_d^r(\mathscr{T})$.

Proof. By construction, the $l_\mathbf{i}$ are linearly independent, and it follows that the dimension of $S_d^r(\mathscr{T})$ is at least $|\mathscr{D}|$. By Lemma 7 we know that $|\mathscr{D}|$ is an upper bound for the dimension of $S_d^r(\mathscr{T})$, and the result follows. ∎

For numerical applications, it is important that the supports of basis elements be small. Any spline in $S_d^r(\mathscr{T})$ which does not vanish at a vertex V must have support containing at least the star of that vertex. Thus, any basis for $S_d^r(\mathscr{T})$ must contain elements with supports on such sets. This observation motivates the following definition:

DEFINITION 9. A basis for the spline space $S_d^r(\mathscr{T})$ is said to be *minimally supported* provided that each basis element has support contained in star(V) for some vertex V.

We shall see below that with sufficient care in the choice of a minimal determining set for $S_d^r(\mathscr{T})$, it is sometimes possible to arrange that the cardinal splines $l_\mathbf{i}$ form a minimally supported basis for $S_d^r(\mathscr{T})$.

4. THE SPACE $S_d^0(\mathscr{T})$

In this section we give a formula for the exact dimension of $S_d^0(\mathscr{T})$, and explicitly construct a minimally supported basis.

THEOREM 10. *For all $d \geqslant 1$,*

$$\dim S_d^0(\mathscr{T}) = \sum_{\sigma \in \mathscr{S}} \binom{d-1}{\dim \sigma} \tag{22}$$

Moreover, the cardinal splines

$$l_{\mathbf{i}}(x) = \mathbf{b}^{\mathbf{i}}(x), \qquad \mathbf{i} \in \mathscr{D} = I_d \tag{23}$$

provide a basis of minimally supported splines.

Proof. It is clear that the splines defined in (23) belong to $S_d^0(\mathscr{T})$ and satisfy (21). Since clearly $\mathscr{D} = I_d$ is a determining set for $S_d^0(\mathscr{T})$, the result follows immediately from Lemma 8 and Lemma 5. ∎

EXAMPLE 11. In the bivariate case ($k = 2$), the simplices in $\mathscr{S}$ are vertices, edges, and triangles. There is one domain point at each vertex, $d-1$ domain points in the interior of each edge, and $\binom{d-1}{2}$ domain points in the interior of each triangle. Letting B be the number of boundary vertices, I the number of interior vertices, E the number of edges, and F the number of triangles, we get

$$\dim S_d^0 = N + (d-1)E + \binom{d-1}{2} F. \tag{24}$$

Now using the Euler relations $F = B + 2I - 2$ and $E = 2B + 3I - 3$, we get

$$\dim S_d^0 = \alpha B + \beta I + \gamma, \tag{25}$$

where

$$\alpha = \frac{d^2+d}{2}, \qquad \beta = d^2, \qquad \text{and} \qquad \gamma = -d^2 + 1. \tag{26}$$

This is a special case of a formula given in [2, 17].

EXAMPLE 12. In the trivariate case ($k = 3$), the number of tetrahedra is not determined uniquely by the point set $\mathscr{V}$. Assume the triangulation can be built by adding one tetrahedron at a time, joining it on precisely 1, 2, or 3 facets to the growing triangulation (see [3] for a discussion of this assumption). Let a_μ, $\mu = 1, 2, 3$, denote the number of times that a tetrahedron was joined at precisely μ faces. Letting N be the number of vertices, E the number of edges, F the number of triangular facets, and T the number of tetrahedra, it turns out that

$$\begin{aligned} N &= 4 + a_1, \qquad E = 6 + 3a_1 + a_2, \\ F &= 4 + 3a_1 + 2a_2 + a_3, \qquad T = 1 + a_1 + a_2 + a_3 \end{aligned} \tag{27}$$

and

$$\dim S_d^0(\Omega) = \rho_0 + \sum_{\mu=1}^{3} \rho_\mu a_\mu, \tag{28}$$

where

$$\rho_\mu = \binom{d+3-\mu}{3}, \qquad \mu = 0, 1, 2, 3. \tag{29}$$

This is also a special case of a formula in [2].

5. Smoothness Conditions

To analyze $S_d^r(\mathscr{T})$ where $r>0$, we need to take account of smoothness conditions between the polynomial pieces. We begin with a function $s \in S_d^0(\mathscr{T})$, and give conditions for it to belong to $S_d^r(\mathscr{T})$ for $r>0$. Let $\tau \in \mathscr{S}_{k-1}$ be an *interior* facet shared by the two simplices $\tau_1, \tau_2 \in \mathscr{T}$. Let the vertices V and $\bar{V}$ be defined by

$$\tau_1 = \operatorname{conv}(V \cup \langle \tau \rangle) \qquad \text{and} \qquad \tau_2 = \operatorname{conv}(\bar{V} \cup \langle \tau \rangle). \tag{30}$$

Furthermore, let $\mathbf{a} = [a_W]_{W \in \mathscr{V}} \in \mathbb{R}^N$ be the vector of generalized barycentric coordinates of V with respect to τ_2, i.e.,

$$V = \sum_{W \in \langle \tau_2 \rangle} a_W W, \qquad \sum_{W \in \langle \tau_2 \rangle} a_W = 1, \qquad a_W = 0 \qquad \forall W \notin \langle \tau_2 \rangle. \tag{31}$$

Let $\mathbf{e}_V = [e_W]_{W \in \mathscr{V}} \in \mathbb{N}^N$ be defined by $e_W = \delta_{VW}$. Then it is well-known (see [8]) that $s \in S_d^0(\mathscr{T})$ is r times differentiable everywhere on τ iff

$$c_{\mathbf{i}} = \sum_{\substack{|\mathbf{j}| = \rho \\ P_{\mathbf{i} - \rho \mathbf{e}_V + \mathbf{j}} \in \tau_2}} c_{\mathbf{i} - \rho \mathbf{e}_V + \mathbf{j}} \, \mathbf{a}^{\mathbf{j}} \tag{32}$$

for all $\rho = 1, 2, \ldots, r$ and $\mathbf{i} \in I_d$ such that $i_V = \rho$ and $P_{\mathbf{i}} \in \tau_1$.

Equation (32) describes a smoothness condition of order ρ. We say that the condition is *associated with* the index $\mathbf{i}$ or with the domain point $P_{\mathbf{i}}$. These smoothness conditions involve Bézier coefficients in two adjoining k-simplices. These conditions can chain together, resulting in connections between coefficients in several different k-simplices. To analyze these conditions, we want to localize these interconnections. This can be accomplished by appropriately dividing up the index set (or domain set) as was done in the bivariate case (cf. [4–6, 10–12]). In the remainder of this section we assume that $d > r2^k$.

For each simplex $\sigma \in \mathscr{S}$ of dimension less than k, let

$$\mathscr{B}(\sigma) = \left\{ \mathbf{i} \in I_d \colon \sum_{V \in \langle \sigma \rangle} i_V \geq d - r2^{k - \dim \sigma - 1} \right\} \Big\backslash \bigcup_{\tau \text{ face of } \sigma} \mathscr{B}(\tau). \tag{33}$$

This is a recursive definition; one takes first simplices of dimension 0, then those of dimension 1, etc. Since

$$\sum_{V \in \langle\sigma\rangle} i_V + \sum_{V \in \langle\sigma\rangle'} i_V = d, \tag{34}$$

it is clear that we can also write

$$\mathscr{B}(\sigma) = \{\mathbf{i} \in I_d : d(P_{\mathbf{i}}, \sigma) \leqslant r2^{k-\dim(\sigma)-1}\} \Big\backslash \bigcup_{\tau \text{ face of } \sigma} \mathscr{B}(\tau). \tag{35}$$

If σ is a k-dimensional simplex in $\mathscr{T}$, we define

$$\mathscr{B}(\sigma) = \{\mathbf{i} \in I_d : P_{\mathbf{i}} \in \sigma\} \Big\backslash \bigcup_{\tau \text{ face of } \sigma} \mathscr{B}(\tau). \tag{36}$$

Let

$$\mathscr{B}^*(\sigma) = \{P_{\mathbf{i}} : \mathbf{i} \in \mathscr{B}(\sigma)\}. \tag{37}$$

Then the set $\mathscr{B}^*(\sigma)$ consists of domain points which are at a distance at most $r2^{k-\dim(\sigma)-1}$ from σ, and which do not lie in any of the sets $\mathscr{B}^*(\tau)$ for any subsimplex τ of σ. If σ is a k-simplex, then $\mathscr{B}^*(\sigma)$ consists of domain points in σ which are at a distance of r or more from any face of σ.

EXAMPLE 13. Consider the special case $k=3$, $r=1$, $d=9$, $N=4$. Thus there is a single tetrahedron and a single nonic polynomial with 220 coefficients. These are divided up as follows: For each of the four vertices there are 35 domain points within a distance of 4 of the vertex. For each of the six edges, there are 8 domain points with a distance of at most 2 from the edge (but not in the vertex sets). For each triangular face σ of the tetrahedron, there are 7 domain points in $\mathscr{B}^*(\sigma)$ at a distance of at most 1 from the face. Finally, there are 4 domain points inside the tetrahedron which are not in any of the previous sets.

We claim that the sets defined in (33) form a partition of I_d. Clearly, the union of the sets $\mathscr{B}^*(\sigma)$ includes all of the domain points; i.e.,

$$I_d = \bigcup_{\sigma \in \mathscr{S}} \mathscr{B}(\sigma). \tag{38}$$

It remains to show that the sets in (33) are disjoint.

LEMMA 14. *For all* $\sigma, \tau \in \mathscr{S}$,

$$\sigma \neq \tau \quad \Rightarrow \quad \mathscr{B}(\sigma) \cap \mathscr{B}(\tau) = \varnothing. \tag{39}$$

Proof. Suppose there is a domain index $\mathbf{i} \in \mathscr{B}(\sigma) \cap \mathscr{B}(\tau)$, for two simplices $\sigma, \tau \in \mathscr{S}$ such that $\dim \sigma \geqslant \dim \tau$, and $\tau \not\subset \sigma$. Thus

$$\sum_{V \in \langle \sigma \rangle} i_V \geqslant d - r2^{k - \dim \sigma - 1}, \quad \sum_{V \in \langle \tau \rangle} i_V \geqslant d - r2^{k - \dim \tau - 1}, \tag{40}$$

implying

$$\sum_{V \in \langle \sigma \rangle} i_V + \sum_{V \in \langle \tau \rangle} i_V \geqslant 2d - r2^{k - \dim \sigma - 1} - r2^{k - \dim \tau - 1}. \tag{41}$$

Also,

$$\sum_{V \in \langle \sigma \rangle} i_V + \sum_{V \in \langle \tau \rangle} i_V = \sum_{V \in \langle \sigma \cap \tau \rangle} i_V + \sum_{V \in \langle \sigma \rangle \cup \langle \tau \rangle} i_V. \tag{42}$$

Combining, and using $\sum_{V \in \langle \sigma \rangle \cup \langle \tau \rangle} i_V \leqslant d$, we obtain

$$\sum_{V \in \langle \sigma \cap \tau \rangle} i_V \geqslant d - r2^{k - \dim \tau}. \tag{43}$$

Thus there exists a face $\tilde{\tau}$ of τ such that $\mathbf{i} \in \mathscr{B}(\tilde{\tau})$, which is a contradiction. ∎

The following lemma shows how the domain points involved in a smoothness condition lying in one simplex interact with domain points lying in lower order simplices.

LEMMA 15. *Let $d > r2^k$ and let $\sigma \in \mathscr{S}$ be a given simplex. Suppose $\mathbf{i} \in \mathscr{B}(\sigma)$. Then for each domain index $\mathbf{i} - \rho \mathbf{e}_V + \mathbf{j}$ on the right hand side of the smoothness condition* (32) *there exists a simplex $\omega \subset \sigma$ such that $\mathbf{i} - \rho \mathbf{e}_V + \mathbf{j} \in \mathscr{B}(\omega)$.*

Proof. Fix $\mathbf{i} \in \mathscr{B}(\sigma)$ and let τ, ρ, and V be as in (32). We have to show that for all $\mathbf{j}$ as in (32), $\mathbf{i} - \rho \mathbf{e}_V + \mathbf{j} \in \mathscr{B}(\omega)$ for some $\omega \subset \sigma$. Since $\mathbf{i} \in \mathscr{B}(\sigma)$, we have $d(P_{\mathbf{i}}, \sigma) \leqslant r2^{k - \dim \sigma - 1}$. Now the index $\mathbf{i} - \rho \mathbf{e}_V + \mathbf{j}$ is obtained from $\mathbf{i}$ by reducing the coordinate corresponding to V by ρ, and by increasing other components by at most $|\mathbf{j}| \leqslant \rho$. By the definition of distance and the fact that $V \notin \sigma$, we conclude that

$$d(P_{\mathbf{i} - \rho \mathbf{e}_V + \mathbf{j}}, \sigma) \leqslant d(P_{\mathbf{i}}, \sigma) \leqslant r2^{k - \dim \sigma - 1}, \tag{44}$$

and the result follows. ∎

EXAMPLE 16. Let $k=3$. Then the smoothness conditions associated with domain points lying in the disk $\mathscr{B}^*(V)$ around a vertex involve only points in that disk. The smoothness conditions associated with domain points lying in the set $\mathscr{B}^*(E)$ for an edge E involve points in E and in the two disks $\mathscr{B}^*(V_1)$ and $\mathscr{B}^*(V_2)$ around the vertices V_1 and V_2 which form the ends of E. The smoothness conditions associated with domain points lying in the set $\mathscr{B}^*(F)$ for a (triangular) face F involve domain points lying in the three sets $\mathscr{B}^*(E_1)$, $\mathscr{B}^*(E_2)$ and $\mathscr{B}^*(E_3)$, where E_1, E_2, E_3 are the edges of the face, and in the three sets $\mathscr{B}^*(V_1)$, $\mathscr{B}^*(V_2)$, $\mathscr{B}^*(V_3)$, where V_1, V_2, V_3 are the three vertices of F. Finally, for a tetrahedron T, the set $\mathscr{B}^*(T)$ consists only of points in T which are not involved in any smoothness conditions (being a distance of more than r from every face).

6. Trivariate Splines on a Special Class of Partitions

In the following section we present the main result of the paper concerning the existence of local bases for trivariate spline spaces defined over a tetrahedral partition. In preparation for proving this result, in this section we discuss trivariate splines on special kinds of tetrahedral partitions, called "oranges" in [21].

DEFINITION 17. Let Ω be the union of a set $\mathcal{O}$ of tetrahedra $T^{[1]}, ..., T^{[n]}$ which share a common interior edge. We call such a partition an *orange*.

Given an orange $\mathcal{O}$ of Ω, we may assume that the common edge is the z axis. Let v_T and v_B be the top and bottom vertices of Ω. Then the remaining vertices can be ordered counterclockwise according to their projections (x_i, y_i) into the (x, y) plane. We number them as $v_1, ..., v_n$, and suppose that none of them lie on the y axis. Suppose that the tetrahedra are numbered so that $T^{[i]}$ has vertices v_T, v_B, v_i, v_{i+1} (where $v_{n+1} := v_1$). The face separating the tetrahedra $T^{[i-1]}$ and $T^{[i]}$ is a vertical plane with the equation $y+\sigma_i x=0$, where $\sigma_i = -y_i/x_i$.

LEMMA 18. *Every $s \in \mathscr{S}_d^r(\mathcal{O})$ can be written in the form*

$$s(x, y, z) = p(x, y, z) + \sum_{i=1}^{n} \sum_{j=1}^{d-r} \sum_{k=0}^{j-1} \sum_{l=0}^{j-k-1} a_{jkl}^{[i]} \phi_{jkl}^{[i]}(x, y, z), \tag{45}$$

where

$$\phi_{jkl}^{[i]}(x, y, z) = \begin{cases} z^k x^l (y+\sigma_i x)^{r+j-k-l}, & (x, y, z) \in T^{[\mu]},\ \mu \geq i \\ 0, & \textit{otherwise}, \end{cases} \tag{46}$$

and where p is a polynomial of total degree d in (x, y, z). *Moreover, the coefficients* $a_{jkl}^{[i]}$ *must satisfy a homogeneous linear system of equations of the form* $Aa = 0$, *where* $a = (a_{d-r}, ..., a_1)^T$, *with* $a_j = (a_{d-r,d-r-j}, ..., a_{j,0})^T$,

$$a_{j+k,k} = (a_{j+k,k,0}^{[1]}, ..., a_{j+k,k,j-1}^{[1]}, ..., a_{j+k,k,0}^{[n]}, ..., a_{j+k,k,j-1}^{[n]})^T, \tag{47}$$

and where

$$A = \begin{bmatrix} A_{d-r} & & \\ & \ddots & \\ & & A_1 \end{bmatrix}, \tag{48}$$

$$A_j = \begin{bmatrix} A_{d-r,d-r-j} & & \\ & \ddots & \\ & & A_{j,0} \end{bmatrix}, \tag{49}$$

with $A_{k+j,k} = (A_{k+j,k}^{[1]}, ..., A_{k+j,k}^{[n]})$ *and*

$$A_{k+j,k}^{[i]} = \begin{bmatrix} 1 & 0 & & \\ \binom{r+j}{1}\sigma_i & 1 & & \\ \binom{r+j}{2}\sigma_i^2 & \binom{r+j-1}{1}\sigma_i & \ddots & \\ \vdots & \vdots & \vdots & \\ & & & 1 \\ & & & \binom{r+1}{1}\sigma_i \\ & & & \vdots \\ \binom{r+j}{r+j}\sigma_i^{r+j} & \binom{r+j-1}{r+j-1}\sigma_i^{r+j-1} & \cdots & \binom{r+1}{r+1}\sigma_i^{r+1} \end{bmatrix}. \tag{50}$$

Proof. The statement that s can be represented as stated is a direct analog of the two-variable result in [15] (see also [17, 18]). By a standard algebraic argument, the difference between the polynomial pieces p_i and p_{i+1} which share the face with equation $y + \sigma_i x = 0$ must be a linear combination of polynomials each of which contains the factor $(y + \sigma_i x)^{r+1}$. The space of all such polynomials (of total degree d) is spanned by the polynomials $\phi_{jkl}^{[i]}$ with $1 \leqslant j \leqslant d-r$, $0 \leqslant k \leqslant j-1$, and $0 \leqslant l \leqslant j-k-1$.

To show that the coefficients must satisfy the system $Aa = 0$, we observe that after crossing the nth face, the expression for s must agree with the original polynomial; i.e., the coefficients of the various powers of x, y, z of the difference $s-p$ must all be zero. The conditions corresponding to the monomials $z^k y^{r+j}$, $z^k x y^{r+j-1}, ..., z^k x^{r+j}$ lead to the equations $A_{j+k,k} a_{j+k,k} = 0$. Assembling these equations for $1 \leqslant j \leqslant d-r$ and $0 \leqslant k \leqslant d-r-j$ leads to the system $Aa = 0$. ∎

THEOREM 19. *For all $0 \leqslant r < d$, the dimension of $\mathscr{S}^r_d(\mathcal{C})$ is given by*

$$\binom{d+3}{3} + n\binom{d-r+2}{3} - (d+3)\binom{d-r+1}{2} + 2\binom{d-r+2}{3} + \sigma, \tag{51}$$

where

$$\sigma = \sum_{j=1}^{d-r} (d-r-j+1)(r+j+1-je)_+, \tag{52}$$

and e is the number of distinct numbers in the sequence $|\sigma_1|, ..., |\sigma_n|$.

We remark that $\dim \mathscr{S}^r_d(\mathcal{C}) = \sum_{\delta=0}^{d} S^r_\delta(C)$ where C is the two dimensional vertex star (i.e., the union of all triangles sharing the interior vertex) obtained by projecting $\mathcal{C}$ along the interior edge, c.f., [21].

Proof. The dimension of the space of polynomials in (x, y, z) is $\binom{d+3}{3}$. The number of linear independent nonpolynomial elements in $\mathscr{S}^r_d(\mathcal{C})$ is equal to the number of linearly independent solutions of the homogeneous system $Aa = 0$. This system consists of E equations in U unknowns, where

$$E = \sum_{j=1}^{d-r} (d-r-j+1)(r+j+1) = \sum_{j=1}^{d-r} j(d-j+2) = (d+3)\binom{d-r+1}{2} - 2\binom{d-r+2}{3} \tag{53}$$

and

$$U = \sum_{i=1}^{n} \sum_{j=1}^{d-r} \sum_{k=0}^{j-1} \sum_{l=0}^{j-k-1} 1 = n\sum_{j=1}^{d-r} (d-r-j+1)j = n\binom{d-r+2}{3}. \tag{54}$$

Now the number of linearly independent solutions of $Aa = 0$ is given by $U - R$, where in view of the band structure of A,

$$R = \operatorname{rank}(A) = \sum_{j=1}^{d-r} \operatorname{rank}(A_j) = \sum_{j=1}^{d-r} \sum_{k=0}^{d-r-j} \operatorname{rank}(A_{j+k,k}). \tag{55}$$

Thus, to calculate R, we need only consider the matrices $A^{[i]}_{j+k,k}$, which are of size $r+j+1$ by j. It was shown in [15, 18] that

$$\operatorname{rank}(A_{j+k,k}) = (r+j+1) - (r+j+1-je)_+. \tag{56}$$

The result follows. ∎

Following [18], we now show how to explicitly choose a minimal determining set of indices for the spline space $\mathscr{S}^r_d(\mathcal{O})$ in Theorem 19. First, we need the analog of Lemma 3.1 of [18]. Suppose that the ordinates for tetrahedron $T^{[i]}$ are given by

$$c^{[i]}_{jklm}, \qquad j+k+l+m=d, \tag{57}$$

with the vertices of $T^{[i]}$ are v_T, v_B, v_i, and v_{i+1} in order.

LEMMA 20. *Suppose that $s \in \mathscr{S}^r_d(\mathcal{O})$ is such that $s \equiv 0$ on the tetrahedron $T^{[i-1]}$. Let*

$$a^{[i]} = (a^{[i]}_{d-r}, \ldots, a^{[i]}_1)^T, \qquad \textit{where} \quad a^{[i]}_p = (a^{[i]}_{d-r,d-r-p}, \ldots, a^{[i]}_{p,0})^T \tag{58}$$

with

$$a^{[i]}_{p+q,q} = (a^{[i]}_{p+q,q,0}, \ldots, a^{[i]}_{p+q,q,p-1})^T. \tag{59}$$

In addition, let

$$c^{[i]} = (c^{[i]}_d, \ldots, c^{[i]}_{r+1})^T, \qquad \textit{where} \quad c^{[i]}_p = (c^{[i]}_{p,0}, \ldots, c^{[i]}_{p,d-p})^T \tag{60}$$

with

$$c^{[i]}_{p,q} = (c^{[i]}_{d-p-q,q,0,p}, \ldots, c^{[i]}_{d-p-q,q,p-r-1,r+1})^T. \tag{61}$$

Then there exists an upper triangular matrix $U^{[i]}$ with nonzero diagonal entries such that

$$a^{[i]} = U^{[i]} c^{[i]}. \tag{62}$$

Proof. Without loss of generality, we may assume that the vertices of the tetrahedron $T^{[i]}$ are at $v_T = (0, 0, v^3_T)$, $v_B = (0, 0, 0)$, $v_i = (v^1_i, 0, v^3_i)$, and $v_{i+1} = (v^1_{i+1}, v^2_{i+1}, v^3_{i+1})$, and that the common face between tetrahedron $T^{[i-1]}$ and $T^{[i]}$ is the (x, z) plane. Direct calculations show that the barycentric coordinates of a point (x, y, z) in $T^{[i]}$ with respect to the four vertices of $T^{[i]}$ are given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha v_T + \beta v_B + \gamma v_i + \delta v_{i+1}, \tag{63}$$

where

$$
\begin{aligned}
\alpha &= \frac{(v_{i+1}^1 y - v_{i+1}^2 x)\, v_i^3 - v_i^1 v_{i+1}^3\, y + v_i^1 v_{i+1}^2 z}{v_T^3 v_i^1 v_{i+1}^2} \\
\beta &= 1 - \alpha - \gamma - \delta \\
\gamma &= \frac{v_{i+1}^2 x - v_{i+1}^1 y}{v_i^1 v_{i+1}^2} \\
\delta &= \frac{y}{v_{i+1}^2}.
\end{aligned}
\tag{64}
$$

It is well known (cf. [8]) that there is a lower triangular matrix L with nonzero diagonal entries such that $L\phi = \psi$, where

$$\phi = (\phi_{d-r}, ..., \phi_1)^T, \qquad \text{where} \quad \phi_p = (\phi_{d-r,d-r-p}, ..., \phi_{p,0})^T, \tag{65}$$

and

$$\phi_{p+q,q} = (z^q y^{p+r}, ..., z^q x^{p-1} y^{r+1})^T, \tag{66}$$

and where

$$\psi = (\psi_d, ..., \psi_{r+1})^T \qquad \text{with} \quad \psi_p = (\psi_{p,0}, ..., \psi_{p,d-p})^T, \tag{67}$$

and

$$\psi_{p,q} = (\alpha^{d-p-q} \beta^q \delta^p, ..., \alpha^{d-p-q} \beta^q \gamma^{p-r-1} \delta^{r+1})^T. \tag{68}$$

Using this connection, we see that $(c^{[i]})^T \psi = (c^{[i]})^T L\phi = (a^{[i]})^T \phi$. This implies that $a^{[i]} = L^T c^{[i]}$, and hence $U^{[i]} = L^T$ is the desired matrix. ∎

We say that the coefficient $c_{jklm}^{[i]}$ is on a qth ring around the line from v_B to v_T provided that $l + m = q$. We now describe how to choose Bézier coordinates to determine all of the ordinates on such a ring.

THEOREM 21. *Suppose that $s \in \mathcal{S}_{r+j}^r(\mathcal{O})$ on a partition $\mathcal{O}$ as in Definition 17, with $1 \leqslant j$. In addition suppose that all of the Bézier ordinates of s in all rings up to order $r + j - 1$ and all of the ordinates of s in the tetrahedron $T^{[1]}$ are zero. Let Γ_j be the indices of the first*

$$N = nj - (r + j + 1) + (r + j + 1 - je)_+ \tag{69}$$

ordinates in the ordered set

$$\{c_{0,0,r+1,j-1}^{[\mu_1]}, ..., c_{0,0,0,r+j}^{[\mu_1]}, ..., c_{0,0,r+1,j-1}^{[\mu_n]}, ..., c_{0,0,0,r+j}^{[\mu_n]}\}, \tag{70}$$

where the c's are as in Lemma 20, and where $\mu_{n-e+1} < \cdots < \mu_n = n$ are such that the associated faces are pairwise distinct, and where $\mu_1 < \cdots < \mu_{n-e}$ is *a complementary set so that* $\{\mu_1, ..., \mu_n\} = \{1, ..., n\}$. *Then* $\lambda_{\mathbf{i}} s = 0$ *for* $\mathbf{i} \in \Gamma_j$ *implies* $s \equiv 0$.

Proof. First we note that

$$(n-e)j \leqslant nj - (r+j+1) + (r+j+1-je)_+ \leqslant (n-1)j. \tag{71}$$

This assures that in defining Γ_j, all of the indices of ordinates with superscripts $\mu_1, ..., \mu_{n-e}$ are selected, while no index of an ordinate with superscript $\mu_n = n$ is selected.

We apply Lemma 20 to force all but $(r+j+1)-(r+j+1-je)_+$ of the coefficients in the expansion

$$(s-p)(x, y, z) = \sum_{i=1}^{n} \sum_{l=0}^{j-1} a_{j0l}^{[i]} \phi_{j0l}^{[i]}(x, y, z) \tag{72}$$

to be zero (cf. the arguments in [18]). These remaining coefficients are associated with at most e edges with different slopes, and satisfy a homogeneous system of full rank, and thus must be zero. ∎

From the above results, it is clear that to construct a minimal determining set for $\mathcal{S}^r_d(\mathcal{O})$, we can begin with the set of all indices of the points in the first tetrahedron $T^{[1]}$. The smoothness conditions then ensure that all ordinates are determined in the first r rings. Now we apply the above theorem to determine the ordinates for all rings of order $r+1$, then for rings of order $r+2$, etc. Each of the $d-r-j+1$ rings of order $r+j$ can be handled separately. Indeed, the equations corresponding to any ring of order $r+j$ are all the same, and in fact are all equivalent to the equations obtained in considering $\mathcal{S}^r_{r+j}(\mathcal{O})$.

THEOREM 22. *Suppose that* $s \in \mathcal{S}^r_d(\mathcal{O})$ *on an orange* $\mathcal{O}$. *Let* Γ_0 *be the set of all indices corresponding to domain points in the tetrahedron* $T^{[1]}$. *For each* $1 \leqslant j \leqslant d-r$ *and* $0 \leqslant k \leqslant d-r-j$, *let* $\Gamma_{j,k}$ *be the set of indices which determine the kth ring of order* $r+j$. *Then the set*

$$\Gamma = \Gamma_0 \cup \bigcup_{j=1}^{d-r} \bigcup_{k=0}^{d-r-j} \Gamma_{j,k} \tag{73}$$

is a minimal determining set for $\mathcal{S}^r_d(\mathcal{O})$.

Proof. Clearly Γ is a determining set. To show it is minimal, we need only check that its cardinality is equal to the dimension of $\mathcal{S}^r_d(\mathcal{O})$. But

$$\#(\Gamma) = \binom{d+2}{3} + \sum_{j=1}^{d-r} \sum_{k=0}^{d-r-j} (nj - (r+j+1) + (r+j+1-je)_+) \tag{74}$$

which is precisely the number in Theorem 19. ∎

7. The Existence of Local Bases for Trivariate Splines

We now discuss the problem of constructing a minimal determining set for $S_d^r(\mathscr{T})$ in the special case of three variables ($k=3$) and where $d>8r$. The idea is to first find minimal determining subsets for each of the sets $\mathscr{B}(\sigma)$, where σ is a vertex. Then we work on the $\mathscr{B}(\sigma)$ where σ runs through the edges, then through the triangles, and finally through the tetrahedra.

Definition 23. Fix the spline space $S_d^r(\mathscr{T})$. Let $\sigma\in\mathscr{S}$. We call a set $\mathscr{D}(\sigma)\subset\mathscr{B}(\sigma)$ a determining set for $\mathscr{B}(\sigma)$ provided that for all $s\in S_d^r(\mathscr{T})$,

$$c_{\mathbf{i}}=0\ \ \forall\mathbf{i}\in\mathscr{D}(\sigma)\cup\bigcup_{\tau\text{ face of }\sigma}\mathscr{B}(\tau)\quad\Rightarrow\quad c_{\mathbf{i}}=0\ \ \forall\mathbf{i}\in\mathscr{B}(\sigma). \tag{75}$$

The set $\mathscr{D}(\sigma)$ is a minimal determining set of $\mathscr{B}(\sigma)$ if there is no smaller set which works.

The following theorem is the main result of the paper.

Theorem 24. *Let $k=3$, $r>0$, and $d>8r$. For each simplex $\sigma\in\mathscr{S}$ let $\mathscr{D}(\sigma)$ be a minimal determining set for $\mathscr{B}(\sigma)$. Then*

$$\mathscr{D}:=\bigcup_{\sigma\in\mathscr{S}}\mathscr{D}(\sigma) \tag{76}$$

is a minimal determining set for $S_d^r(\Omega)$, and

$$\dim S_d^r(\Omega)=|\mathscr{D}|=\sum_{\sigma\in\mathscr{S}}|\mathscr{D}(\sigma)|. \tag{77}$$

Moreover, for each $\sigma\in\mathscr{S}$ and each $\mathbf{i}\in\mathscr{D}(\sigma)$, there is an associated cardinal spline $l_{\mathbf{i}}$ with support on star(σ). *The collection*

$$\Lambda=\{l_{\mathbf{i}}:\mathbf{i}\in\mathscr{D}\} \tag{78}$$

forms a minimally supported basis for $S_d^r(\mathscr{T})$, and the set of linear functionals

$$\Lambda^*=\{\lambda_{\mathbf{i}}:\mathbf{i}\in\mathscr{D}\} \tag{79}$$

defined as in (15) *form a dual basis in the sense that* (21) *is satisfied.*

Proof. Using induction on $\dim\sigma$ for simplices $\sigma\in\mathscr{S}$, we first establish that $\mathscr{D}$ is a determining set. Let $\sigma\in\mathscr{S}$. If $\dim\sigma=0$, then σ is a vertex, and by Lemma 15 the smoothness conditions (32) only involve domain indices in $\mathscr{B}(\sigma)$ and by assumption all domain indices in $\mathscr{B}(\sigma)\setminus\mathscr{D}(\sigma)$ are determined. If $\dim\sigma>0$ then the smoothness conditions only involve domain indices in $\mathscr{B}(\sigma)$ and $\mathscr{B}(\tau)$ for lower dimensional simplices τ. The domain indices in all sets $\mathscr{D}(\tau)$ are determined by the induction hypothesis. The remaining domain indices in $\mathscr{B}(\sigma)$ are determined by virtue of $\mathscr{D}(\sigma)$ being a determining set. We have established that $\mathscr{D}$ is a determining set, and hence by Lemma 7 that the dimension of $S_d^r(\mathscr{T})$ is bounded above by $|\mathscr{D}|$.

The theorem now follows from Lemma 8 if we can show that for each $\sigma \in \mathscr{S}$ and each $\mathbf{i} \in \mathscr{D}(\sigma)$, there is an associated cardinal spline with support on star(σ) satisfying (21).

Case 1. (σ is a vertex). For given $\mathbf{i} \in \mathscr{D}(\sigma)$, set $c_{\mathbf{i}} = 1$ and $c_{\mathbf{j}} = 0$ for all other $\mathbf{j} \in \mathscr{D}$. Then clearly $l_{\mathbf{i}}$ is uniquely determined to be zero at all indices not belonging to a tretrahedron T with a vertex at σ. In addition, the coefficient of $l_{\mathbf{i}}$ is uniquely determined to be zero at all indices belonging to balls of the form $\mathscr{B}(V)$, where $V \neq \sigma$ is a vertex of $\mathscr{T}$.

Now we show that $l_{\mathbf{i}}$ can be extended to each of the sets $\mathscr{B}(E)$, where E is an edge. We need only consider edges E such that σ is one of the endpoints of E. We can apply the results of Section 6 for oranges. First we show how to extend $l_{\mathbf{i}}$ to the complex of indices C in $\mathscr{B}(E)$ at a distance of $4r+1$ from σ. We start with the index of the point on E, and then proceed to the point lying in C and on the qth ring around E for $q = 1, 2, ..., 2r$. If $q \leqslant r$, these indices are uniquely determined by the r continuity conditions (to be a polynomial of degree r). Suppose $q > r$. There are nq points on this ring, and the associated coefficients must satisfy a total of nr smoothness conditions. This means that there are always some free coefficients, and the function $l_{\mathbf{i}}$ can be extended. This process can be carried out for each $1 \leqslant q \leqslant 2r$, and $l_{\mathbf{i}}$ has been extended to C. This process can then be repeated for the complex at distance $4r+2$ from σ, etc. until all coefficients corresponding to indices of points in $\mathscr{D}(E)$ have been determined. (Note, that some of the complexes farthest from σ will intersect $\mathscr{B}(W)$, where W is the vertex at the other end of E. In this case a number of rings may already be determined to be zero).

Next we have to show that $l_{\mathbf{i}}$ can be further extended to the sets $\mathscr{B}(F)$ where F is a face of a tetrahedron. We are only interested in faces F which have σ as a vertex as we can take $l_{\mathbf{i}}$ to be zero on all other faces. Let F be such a face. Now each of the smoothness conditions across F involves a pyramid of points. In particular, a typical smoothness condition of order p will involve the index $\mathbf{i}$ of a point at distance p from F, and the index of a similar point located symmetrically at a distance p from F on the other side of F. One of these points is always free to choose, and it follows that $l_{\mathbf{i}}$ can be extended to $\mathscr{B}(F)$.

Finally, it is trivial that $l_{\mathbf{i}}$ can be extended to the sets $\mathscr{B}(T)$, where T is a tetrahedron as points in these sets are not involved in any smoothness conditions, and hence we can take $l_{\mathbf{i}}$ to be zero there.

Case 2. (σ is an edge). For given $\mathbf{i} \in \mathscr{D}(\sigma)$, set $c_{\mathbf{i}} = 1$ and $c_{\mathbf{j}} = 0$ for all other $\mathbf{j} \in \mathscr{D}$. Then clearly $l_{\mathbf{i}}$ is uniquely determined on the set $\mathscr{B}(\sigma)$. Moreover, it is uniquely determined to be identically zero on all sets except for those of the form $\mathscr{B}(F)$, where F is a face sharing the edge σ. But $l_{\mathbf{i}}$ can be extended to such a face by the same argument as in Case 1.

Case 3. (σ is a face or a tetrahedron). This case is trivial. ∎

In the bivariate case, it is known (see [11, 12] and also [9]) that minimally supported bases exist not only for all $d \geqslant 4r+1$, but also for all $d \geqslant 3r+2$. It is to be expected that for the general case considered here, minimally supported bases exist not only for all $d \geqslant r2^k+1$, but also for all $d \geqslant r(2^k-1)+k$.

8. The Dimension Problem

Theorem 24 gives a formula for the dimension of $S_d^r(\mathscr{T})$ in terms of the number of indices in the minimal determining sets $D(\sigma)$ for $\mathscr{B}(\sigma)$ in (33) associated with vertices, edges, triangles, and tetrahedra. This localizes the dimension problem. Unfortunately, the analysis of minimal determining sets for the sets $\mathscr{B}(\sigma)$ corresponding to vertices is an extremely difficult problem. In this case, the smoothness conditions (32) for all $\mathbf{i} \in \mathscr{B}(\sigma)$ are identical to the set of equations governing the smoothness of a function $s \in S^r_{r2^k-1}(\text{star}(\sigma))$ and

$$|\mathscr{D}(\sigma)| = \dim S^r_{r2^k-1}(\text{star}(\sigma)). \tag{80}$$

Thus, constructing a minimal determining set for $\mathscr{B}(\sigma)$ is equivalent to dealing with spline spaces defined on a *vertex star*, i.e., the set of k-simplices surrounding a single vertex. The following example shows that we cannot expect to get formulae even for trivariate vertex stars without first completely understanding the bivariate dimension problem for arbitrary degrees and partitions.

Example 25. Consider any two-dimensional triangulation Δ with more than one interior vertex. Lift it to three dimensions by adding a point above the triangulation and connecting that point to each of the vertices of the two-dimensional triangulation. This results in a tetrahedral partition $\mathscr{T}$ such that each tetrahedron has one face in the plane we started with. Now consider a spline on the three-dimensional triangulation $\mathscr{T}$. Because all but one of the points lie in a plane, the smoothness conditions decouple into $d+1$ sets, each of which describes a bivariate spline on the original two-dimensional triangulation. The polynomial degrees of these splines are

TABLE I

S_9^1 on the Trivariate Clough–Tocher Split

Simplex type	Dimension	# Occurences	$\lvert\mathscr{B}(\sigma)\rvert$	$\lvert\mathscr{D}(\sigma)\rvert$	Total
Boundary vertex	0	4	65	43	172
Interior vertex	0	1	69	38	38
Boundary edge	1	6	13	10	60
Interior edge	1	4	15	8	32
Boundary face	2	4	7	7	28
Interior face	2	6	13	7	42
Tetrahedron	3	4	4	4	16
Total			$589 = \dim S_9^0$		$388 = \dim S_9^1$

$0, 1, \ldots, d$. Thus if we can describe a minimal determining set for this three-dimensional spline space, then we would automatically have a minimal determining set for each of the spaces of bivariate splines $S_i^r(\Delta)$, $i = 1, \ldots, d$. But, in general, dimension formulae have only been obtained for $1 \leqslant i \leqslant r$ and for $i \geqslant 3r+2$. Moreover, it has been shown that the dimension of $\mathscr{S}_{2r}^r(\Delta)$ may depend not only on the topology of the triangulation Δ, but also on the exact geometry (cf. [10]).

For a given tetrahedral partition $\mathscr{T}$, it may be possible to analyze the sets $\mathscr{B}(\sigma)$ associated with vertices σ directly. We now give two examples where this has been done using a computer algebra system.

EXAMPLE 26. Let Ω be a tetrahedron, and let $\mathscr{T}$ be the triangulation which results when we take the trivariate Clough–Tocher split of Ω about its centroid into four subtetrahedra. In this case the dimensions of $\mathscr{S}_9^0(\mathscr{T})$ and $\mathscr{S}_9^1(\mathscr{T})$ are 589 and 388, respectively. Table I shows the number of

TABLE II

S_{17}^2 on the Trivariate Clough–Tocher Split

Simplex type	Dimension	# Occurences	$\lvert\mathscr{B}(\sigma)\rvert$	$\lvert\mathscr{D}(\sigma)\rvert$	Total
Boundary vertex	0	4	369	220	880
Interior vertex	0	1	425	199	199
Boundary edge	1	6	70	51	306
Interior edge	1	4	90	44	176
Boundary face	2	4	46	46	184
Interior face	2	6	86	46	276
Tetrahedron	3	4	56	56	224
Total			$3605 = \dim S_{17}^0$		$2245 = \dim S_{17}^2$

indices in the various subsets $\mathscr{B}(\sigma)$, and the numbers in the corresponding minimal determining sets $\mathscr{D}(\sigma)$. Similarly, the dimensions of $\mathscr{S}_{17}^{0}(\mathscr{T})$ and $\mathscr{S}_{17}^{1}(\mathscr{T})$ are 3605 and 2245, respectively; see Table II.

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